The Propagation of Groups of Waves in Dispersive Media, with Application to Waves on Water produced by a Travelling Disturbance.

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CONTENTS.

Section		Page
1.	Introduction	398
2.	Definition of Simple Group	399
3.	The Fourier Integral regarded as a Collection of Groups	400
	(*a*) Damped Harmonic Wave-train	401
	(*b*) Interrupted Simple Wave-train	402
4.	Features of the Integrals Involved	403
5.	Initial Line Displacement on Deep Water	404
6.	Initial Displacement of Finite Breadth	406
7.	Limited Train of Simple Oscillations	407
8.	Initial Impulse on Deep Water	410
9.	Moving Line Impulse on Deep Water	411
10.	Capillary Surface Waves	413
11.	Water Waves due to Gravity and Capillarity	415
12.	Surface Waves in Two Dimensions	415
13.	Point Impulse Travelling over Deep Water	417
	(*a*) The Transverse Wave System	418
	(*b*) The Diverging Wave System	420
	(*c*) The Line of Cusps	421
14.	Point Impulse for Different Media	424
15.	Point Impulse Moving on Water of Finite Depth	426

§ 1. *Introduction.*

The object of this paper is to illustrate the main features of wave propagation in dispersive media. In the case of surface waves on deep water it has been remarked that the earlier investigators considered the more difficult problem of the propagation of an arbitrary initial disturbance as expressed by a Fourier integral, ignoring the simpler theory developed subsequently by considering the propagation of a single element of their integrals, namely, an unending train of simple harmonic waves. The point of view on which stress is laid here consists of a return to the Fourier integral, with the idea that the element of disturbance is not a simple harmonic wave-train, but a simple group, an aggregate of simple wave-trains clustering around a given central period. In many cases it is then possible to select from the integral the few simple groups that are important, and hence to isolate the chief regular features, if any, in the phenomena.

In certain of the following sections well-known results appear; the aim has been to develop these from the present point of view, and so illustrate the dependence of the phenomena upon the character of the velocity function. In the other sections it is hoped that progress has been made in the theory of the propagation of an arbitrary initial group of waves, and also of the character of the wave pattern diverging from a point impulse travelling on the surface.

§ 2. *Definition of Simple Group.*

We have to consider the transmission of disturbances in a medium for which the velocity of propagation of homogeneous simple harmonic wave-trains is a definite function of the wave-length. The kinematically simplest group of waves is composed of only two simple trains, of wave-lengths λ, λ', differing by an infinitesimal amount $d\lambda$; then with the usual approximation we have for the combined effect

$$y = \mathrm{A}\cos\frac{2\pi}{\lambda}(x-\mathrm{V}t)+\mathrm{A}\cos\frac{2\pi}{\lambda'}(x-\mathrm{V}'t)$$

$$= 2\mathrm{A}\cos\frac{\pi d\lambda}{\lambda^2}(x-\mathrm{U}t)\cos\frac{2\pi}{\lambda}(x-\mathrm{V}t), \qquad (1)$$

where
$$\mathrm{U} = \mathrm{V}-\lambda\frac{d\mathrm{V}}{d\lambda}. \qquad (2)$$

The expression (1) may be regarded as representing at any instant a train of wave-length λ, whose amplitude varies slowly with x according to the first cosine factor. Thus it does not represent a form which moves forward unchanged; but it has a certain periodic quality, for the form at any given instant is repeated after equal intervals of time $\lambda/(\mathrm{V}-\mathrm{U})$, being displaced forward through equal distances $\lambda\mathrm{U}/(\mathrm{V}-\mathrm{U})$. The ratio of these quantities, namely U, is called the group-velocity. It has also the following significance: in the neighbourhood of an observer travelling with velocity U the disturbance continues to be approximately a train of simple harmonic waves of length λ.

The most general simple, or elementary, group may be defined in the following manner. Let the central form be a simple harmonic wave of length $2\pi/\kappa_0$, and let the other members be similar waves whose amplitude, wave-length, and velocity differ but slightly from the central type; then, with similar approximation, we have

$$y = \Sigma\mathrm{A}\cos\{\kappa(x-\mathrm{V}t)+\alpha\}$$

$$= \Sigma\mathrm{A}_n\cos\{\kappa_0(x-\mathrm{V}_0t)+(x-\mathrm{U}_0t)\,\delta\kappa_n+\alpha\}. \qquad (3)$$

The range of values of κ being infinitesimal, the group as a whole may be written, as in the previous case, in the form

$$y = \phi(x - \text{U}_0 t)\cos\{\kappa_0(x - \text{V}_0 t) + \beta\}, \tag{4}$$

where ϕ is a slowly varying function; and the group-velocity U_0 is given by

$$\text{U}_0 = \frac{d}{d\kappa_0}(\kappa_0 \text{V}_0). \tag{5}$$

The group, to an observer travelling with velocity U_0, appears as consisting of approximately simple waves of length $2\pi/\kappa_0$. The simple group is, in fact, propagated as an approximately homogeneous simple wave-train; the importance of the group-velocity lies in the fact that any slight departure from homogeneity on a simple wave-train, due to local variation of amplitude or phase, is propagated with the velocity U.

§ 3. *The Fourier Integral regarded as a Collection of Groups.*

An arbitrary disturbance can, in general, be analysed by Fourier's method into a collection of simple wave-trains ranging over all possible values of κ; thus after a time t the disturbance will be given by an expression of the type

$$\int_0^\infty \phi(\kappa)\cos\kappa(x - \text{V}t + \alpha)\,d\kappa, \tag{6}$$

where V is a given function of κ.

The method adopted with these integrals is based on Lord Kelvin's* treatment of the case, in which the amplitude factor $\phi(\kappa)$ is a constant, so that

$$y = \int_0^\infty \cos\kappa(x - \text{V}t)\,d\kappa.$$

An integral solution of this kind is constructed to represent the subsequent effect of an initial disturbance which is infinitely intense, and concentrated in a line through the origin; Lord Kelvin's process gives an approximate evaluation suitable for times and places such that $x - \text{V}t$ is large, and the argument may be stated in the following manner:—

In the dispersive medium the wave-trains included in each differential element of the varying period are mutually destructive, except when they are in the same phase and so cumulative for the time under consideration, this being when the argument of the undulation is stationary in value. Thus each differential element as regards period, in the Fourier integral, represents a disturbance which is very slight except around a certain point which itself changes with the time.

Now if we apply this method to the more general integral (6), we obtain an expression for the total disturbance, attending only to its prominent features and neglecting the rest, provided we assume the change of the amplitude factor $\phi(\kappa)$ to be gradual. On this hypothesis the resulting expression contains the amplitude of the component trains simply as a factor; and in this way the trains for which it is a maximum show predominantly in the formula, which exhibits the main features of the disturbance as they arise from place to place through cumulation of synchronous component trains.

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* Sir W. Thomson, 'Roy. Soc. Proc.,' vol. 42, p. 80 (1887).

The argument shows that in some respects the integral (6) may be more conveniently regarded as a collection of travelling groups instead of simple wave-trains; when $\phi(\kappa)$ is a slowly varying function, the groups will be simple groups of the type (3). The limitations within which this is the case will appear from the subsequent analysis; one method of procedure would be graphical: to take a graph of the fluctuating factor and see that the other factor, which is taken constant, does not vary much within the range that is important for the integral.

In the cases we shall examine, the effect is due to a limited initial disturbance, and the salient features are due to the circumstance that $\phi(\kappa)$ has well-defined maxima; thus the prominent part of the effect can be expressed in the form of simple groups belonging to the neighbourhood of the maxima.

Before considering in detail special cases with assigned forms of the velocity function V, two illustrations of interest may be mentioned.

(a) *Damped harmonic wave-train.*—If $f(x)$ is a function satisfying the conditions for the Fourier transformation, we have

$$f(x) = \frac{1}{\pi}\int_0^\infty d\kappa \int_{-\infty}^\infty f(\omega)\cos\kappa(\omega - x)\,d\omega.$$

For an even function of x, this gives

$$f(x) = \frac{2}{\pi}\int_0^\infty \phi(\kappa)\cos\kappa x\,d\kappa, \quad \text{where} \quad \phi(\kappa) = \int_0^\infty f(\omega)\cos\kappa\omega\,d\omega. \tag{7}$$

Now let $f(x)$ be an even function of x, defined for all values, and such that it is equal to $e^{-\mu x}\cos\kappa' x$ for x positive; then we find

$$2\phi(\kappa) = \frac{\mu}{\mu^2 + (\kappa - \kappa')^2} + \frac{\mu}{\mu^2 + (\kappa + \kappa')^2}. \tag{8}$$

Consider this function $f(x)$ as the initial value of a disturbance y which occurs in a dispersive medium; then the value of y at any time can be expressed, in general, by

$$y = \mathrm{A}\int_0^\infty \phi(\kappa)\cos\kappa(x - \mathrm{V}t)\,d\kappa + \mathrm{B}\int_0^\infty \phi(\kappa)\cos\kappa(x + \mathrm{V}t)\,d\kappa, \tag{9}$$

where A, B are constants which need not be specified further in the present connection.

These integrals are of the type (6), and represent infinite wave-trains travelling in the positive and negative directions respectively. We see from (8) that when μ is small, the amplitude factor $\phi(\kappa)$ consists practically of a single well-defined peak in the neighbourhood of the value κ'. Hence, when the damping coefficient μ is small, the wave-trains in question may be considered as travelling in the form of a group κ' of unchanging waves of this specified structure.

This example serves to illustrate the propagation of a very long train of simple harmonic waves subsiding as they travel owing to a small damping coefficient, and is of interest in connection with Lord Rayleigh's general proof that the group-velocity U is the velocity with which energy is being propagated.* A small damping coefficient μ is introduced by him, so that the energy transmitted is determined by the energy dissipated; the argument, which of course loses its meaning if μ is actually zero, shows that when μ is diminished indefinitely the rate of transmission of energy approaches U as a limiting value. Similarly, although the Fourier transformation is inapplicable when μ is actually zero, we infer from the above analysis that when μ is diminished indefinitely, the disturbance is representable as a simple group of unchanging waves of definite structure.

(b) *Interrupted simple wave-train.*—Consider an initial disturbance defined by

$$\begin{aligned} f(x) &= 0, \qquad (-d < x < d) \\ &= e^{-\mu x} \sin \kappa' (x-d), \qquad (x > d), \\ &= -e^{\mu x} \sin \kappa' (x+d), \qquad (x < -d). \end{aligned}$$

Then the disturbance is given by an expression of the form (9), in which

$$\begin{aligned} 2\phi(\kappa) &= \int_d^\infty e^{-\mu\omega} \cos \kappa' (\omega - d) \cos \kappa\omega \, d\omega \\ &= \frac{(\kappa+\kappa') \cos \kappa' d - \mu \sin \kappa' d}{\mu^2 + (\kappa+\kappa')^2} + \frac{(\kappa'-\kappa) \cos \kappa' d - \mu \sin \kappa' d}{\mu^2 + (\kappa'-\kappa)^2}. \qquad (10) \end{aligned}$$

Now suppose μ and d very small, so that the initial disturbance approximates to an infinite simple harmonic form with a narrow range of discontinuity; we see that the graph of the amplitude factor $\phi(\kappa)$ is then reduced to a single peak in the vicinity of the value κ'. We infer from this example that a very long simple harmonic wave-train which is interrupted for a short interval is kinematically equivalent to a group of unchanging waves, of definite structure ranging round the value $2\pi/\kappa'$ of the wave-length.

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* Lord Rayleigh, 'Proc. Lond. Math. Soc.,' vol. 9, p. 24 (1877).

§ 4. *Features of the Integrals Involved.*

The integrals we have to consider in such problems are of the type

$$y = \int \phi(u) \cos \{f(u)\}\, du. \tag{11}$$

All such integrals we can treat in the same manner, adopting the method employed by Lord Kelvin for the particular case referred to above (§ 3). This method consists in supposing that $f(u)$ is large, so that the cosine factor is a rapidly varying quantity compared with the first factor; thus, much as in the Fresnel discussion of the diffraction of light waves, the prominent part of the graph of the integral is contained within a small range of u for which $f(u)$ is stationary in value, so that the elements are then cumulative. In other words, we select from (11) the group or groups of terms ranging round values u_0 of u which make

$$f'(u_0) = 0. \tag{12}$$

In such a group of terms we may put

$$f(u) = f(u_0) + \tfrac{1}{2}(u-u_0)^2 f''(u_0).$$

Then if we write σ^2 for $\frac{1}{2}(u-u_0)f''(u_0)$, the contribution of the group to the value of (11) is given by

$$y_0 = \left\{\frac{2}{f''(u_0)}\right\}^{\frac{1}{2}} \int_{-\infty}^{\infty} \phi(u_0) \cos\{f(u_0)+\sigma^2\}\, d\sigma, \tag{13}$$

where the limits of the integral may be in general extended, as in diffraction theory, to $\pm\infty$, provided u_0 does not coincide with either limit of the integral (11), and also provided that $f''(u_0)$ is not zero.

Thus we have, from (13),

$$\begin{aligned} y_0 &= \left\{\frac{\pi}{f''(u_0)}\right\}^{\frac{1}{2}} \phi(u_0)\left[\cos\{f(u_0)\} - \sin\{f(u_0)\}\right] \\ &= \left\{\frac{2\pi}{f''(u_0)}\right\}^{\frac{1}{2}} \phi(u_0) \cos\{f(u_0)+\tfrac{1}{4}\pi\}. \end{aligned} \tag{14}$$

This is the sum of the contributions of the constituents of each group around a central value u_0 given by (12), provided the value u_0 comes within the range of values of u in the integral (11).

If $f''(u_0)$ is negative, the corresponding result may be written

$$y_0 = \left\{\frac{2\pi}{-f''(u_0)}\right\}^{\frac{1}{2}} \phi(u_0) \cos\{f(u_0)-\tfrac{1}{4}\pi\}. \tag{15}$$

We write down for reference the similar pair of results for a group of terms from the integral

$$y = \int \phi(u) \sin\{f(u)\}\, du. \tag{16}$$

If $f''(u_0)$ is positive,

$$y_0 = \left\{\frac{2\pi}{f''(u_0)}\right\}^{\frac{1}{2}} \phi(u_0) \cos\{f(u_0) - \tfrac{1}{4}\pi\}; \tag{17}$$

and if $f''(u_0)$ is negative,

$$y_0 = \left\{\frac{2\pi}{-f''(u_0)}\right\}^{\frac{1}{2}} \phi(u_0) \cos\{f(u_0) + \tfrac{1}{4}\pi\}. \tag{18}$$

The chief form in which such integrals occur is

$$y = \int \phi(\kappa) \cos\kappa(x - \mathrm{V}t)\, d\kappa, \quad \text{where} \quad \mathrm{V} = f(\kappa). \tag{19}$$

The principal groups are given by the values κ_0 such that

$$\frac{d}{d\kappa}\{\kappa(x - \mathrm{V}t)\} = 0, \quad \text{or} \quad \frac{x}{t} = \frac{d}{d\kappa}(\kappa\mathrm{V}) = \mathrm{U}. \tag{20}$$

The aggregate value of the group can be written down from one of the previous forms; if $\partial\mathrm{U}/\partial\kappa$ is negative, we should have

$$y_0 = \left\{\frac{2\pi}{-t\,\partial\mathrm{U}/\partial\kappa}\right\}^{\frac{1}{2}} \phi(\kappa_0) \cos\{\kappa_0(x - \mathrm{V}t) + \tfrac{1}{4}\pi\}. \tag{21}$$

As an illustrative example we may suppose a disturbance y to be given at time t by the expression*

$$y = \int_0^\infty \cos\kappa(x - \mathrm{V}t)\, d\kappa. \tag{22}$$

When $x - \mathrm{V}t$ is large, the elementary waves given by (22) reinforce each other only for the simple groups given by values κ_0 for which the argument of the cosine is stationary, so that

$$x - \mathrm{U}t = 0. \tag{23}$$

This equation (23) defines a velocity U such that to an observer starting from the origin and travelling with this velocity the complex disturbance has the appearance of simple waves of length $2\pi/\kappa_0$. Or again, we may regard (23) as giving the predominant value of κ_0 at any position and time in terms of x and t. The features of the disturbance will depend on the form of the velocity function V; we proceed to consider some special forms.

§ 5. *Initial Line Displacement on Deep Water.*

We consider surface waves on an unlimited sheet of deep water, the only bodily forces being those due to gravity. Let the x-axis be in the undisturbed horizontal surface, and the y-axis be drawn vertically upwards. Let η be the elevation of surface waves of small amplitude with parallel crests and troughs perpendicular to the xy-plane. It can be shown that for an initial displacement given by $\eta = \cos \kappa x$, without initial velocity, the surface form at any subsequent time is given by

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* Lord Kelvin, *loc. cit. ante.*

$$\eta = \cos \kappa \mathrm{V} t \cos \kappa x = \tfrac{1}{2}\{\cos \kappa (x - \mathrm{V}t) + \cos \kappa (x + \mathrm{V}t)\},$$

where $$\mathrm{V} = (g/\kappa)^{\frac{1}{2}}. \tag{24}$$

Let $f(x)$ be any even function of x which can be analysed by Fourier's integral theorem. Then, corresponding to an initial surface displacement $f(x)$, without initial velocity, there is a surface form given at any subsequent time by

$$\eta = \frac{1}{2\pi}\int_0^\infty \phi(\kappa) \cos \kappa (x - \mathrm{V}t)\, d\kappa + \frac{1}{2\pi}\int_0^\infty \phi(\kappa) \cos \kappa (x + \mathrm{V}t)\, d\kappa, \tag{25}$$

where $$\phi(\kappa) = \int_{-\infty}^{\infty} f(\omega) \cos \kappa\omega \, d\omega. \tag{26}$$

If we suppose the initial elevation to be limited practically to a line through the origin and assume that $\int_{-\infty}^{\infty} f(x)\, dx = 1$, so that $\phi(\kappa) = 1$, we can use, as an illustration of the procedure, the form

$$\eta = \frac{1}{2\pi}\int_0^\infty \cos \kappa (x - \mathrm{V}t)\, d\kappa + \frac{1}{2\pi}\int_0^\infty \cos \kappa (x + \mathrm{V}t)\, d\kappa. \tag{27}$$

We select from these integrals the groups which give the chief regular features at large distances from the original disturbance. This cumulative group from the first integral is given for a given position and time by the value of κ for which $\kappa(x - \mathrm{V}t)$ is stationary, where $\mathrm{V} = \sqrt{(g/\kappa)}$, so that

$$\frac{x}{t} = \mathrm{U} = \tfrac{1}{2}\sqrt{\frac{g}{\kappa}};$$

and, similarly, from the second integral by

$$-\frac{x}{t} = \tfrac{1}{2}\sqrt{\frac{g}{\kappa}}.$$

Thus there are symmetrical groups of waves proceeding in the two directions from the origin; for x positive we need only consider the first integral in (27), and for x negative the second integral. Thus the predominant wave-length at a point x at time t is given by

$$\kappa = gt^2/4x^2. \tag{28}$$

Evaluating this predominant group by means of expression (21), we obtain the known result.

$$\eta = \frac{g^{\frac{1}{2}}t}{2\pi^{\frac{1}{2}}x^{\frac{3}{2}}} \cos\left(\frac{gt^2}{4x} - \tfrac{1}{4}\pi\right). \tag{29}$$

At a given position, far enough from the source for the train to be taken as unlimited, this indicates oscillations succeeding each other with continually increasing frequency and amplitude; also if we follow a group of waves with given value of κ the amplitude varies inversely as $t^{\frac{1}{2}}$, or inversely as the square root of x.*

§ 6. *Initial Displacement of Finite Breadth.*

If l is the range within which the initial displacement is sensible, the previous results hold with l/x small; further, as Cauchy showed, $gt^2l/4x^2$ must be small if the function $\phi(\kappa)$ of (26) is to be taken as constant. Prof. Burnside† has obtained approximate equations for the surface form due to certain limited initial displacements not confined to an indefinitely narrow strip. From the present point of view, such results can be recovered simply by selecting from the integrals the more important groups of waves.

(*a*) Let the initial displacement be given by

$$f(x) = c\alpha^2/(\alpha^2+x^2), \qquad (30)$$

where α may be supposed small.

Then $$\phi(\kappa) = \int_{-\infty}^{\infty} \frac{c\alpha^2 \cos \kappa\omega}{\alpha^2+\omega^2} = \pi c\alpha e^{-\alpha\kappa}.$$

Hence from (25) the surface form is

$$\eta = \tfrac{1}{2}c\alpha \int_0^{\infty} e^{-\alpha\kappa} \cos \kappa(x-\mathrm{V}t)\, d\kappa + \tfrac{1}{2}c\alpha \int_0^{\infty} e^{-\alpha\kappa} \cos \kappa(x+\mathrm{V}t)\, d\kappa. \qquad (31)$$

For points at some distance from the range in which the original disturbance was sensible, $e^{-\alpha\kappa}$ varies slowly compared with the cosine term; thus we may consider the integrals as made up of simple groups. For x positive we need only consider the first integral.

The predominant value of κ is thus connected with x and t by the same equation (28) as before. Since the greater amplitudes are associated with the smaller values of κ and these have the greater values of U, it is clear that, at a particular point, the disturbance dies away from its maximum at a slower rate than its growth up to it. Using the previous results we can write down the disturbance involved in the main group form as

$$\eta = c\alpha\pi^{\frac{1}{2}} \frac{g^{\frac{1}{2}}t}{2x^{\frac{3}{2}}} e^{-\frac{gt^2\alpha}{4x^2}} \cos\left(\frac{gt^2}{4x} - \tfrac{1}{4}\pi\right). \qquad (32)$$

The following results can be deduced. The cosine term varies rapidly compared with the other factors, hence we may obtain the maximum by considering the latter alone; it is easily seen that this occurs when

$$x/t = \sqrt{(\tfrac{1}{2}g\alpha)}.$$

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* Lamb, 'Proc. Lond. Math. Soc.' (2), vol. 2, p. 371 (1904).

† W. Burnside, 'Proc. Lond. Math. Soc.,' vol. 20, p. 22 (1888).

Thus the maximum is propagated out with uniform velocity; and we see that in its neighbourhood the predominant wave-length is $4\pi\alpha$.

(b) Let the initial displacement have a constant value A over a range of breadth $2c$, and be zero at all other points; then we have

$$\phi(\kappa) = 2\mathrm{A}\frac{\sin\kappa c}{\kappa}.$$

Hence the surface elevation is

$$\eta = \frac{\mathrm{A}}{\pi}\int_0^\infty \frac{\sin\kappa c}{\kappa}\cos\kappa(x-\mathrm{V}t)\,d\kappa + \frac{\mathrm{A}}{\pi}\int_0^\infty \frac{\sin\kappa c}{\kappa}\cos\kappa(x+\mathrm{V}t)\,d\kappa. \tag{33}$$

With the same argument as before, we consider the value of η at a point as due to the most important of a succession of simple groups, that one, namely, for which the argument is there stationary so that the components reinforce over a considerable range of κ; and we can write down, from the previous results, an expression for this group which is valid at least in the vicinity of the travelling maxima of the disturbance. We have

$$\eta = \frac{4\mathrm{A}}{\pi^{\frac{1}{2}}}\left(\frac{x}{gt^2}\right)^{\frac{1}{2}}\sin\frac{gt^2c}{4x^2}\cos\left(\frac{gt^2}{4x}-\tfrac{1}{4}\pi\right), \tag{34}$$

corresponding to Burnside's result in the paper already cited.

Here we have a succession of maxima given by those of $(x/gt^2)^{\frac{1}{2}}\sin(gt^2/4x^2)$, that is, at times given by $\tan\theta = 2\theta$, where $\theta = gt^2c/4x^2$.

The period of the group that is thus cumulative is different for different localities, and for different times at the same locality; but the accumulation is very prominent only for those times and localities which give a maximum value to the amplitude, which has been graphed for the next example in fig. 1.

The maxima here diminish continually in value, and are propagated each with uniform velocity, namely, the group-velocity corresponding to the predominant wave-length in the neighbourhood.

§ 7. *Limited Train of Simple Oscillations.*

Another interesting example is the case of an initial displacement consisting of a limited length of simple harmonic oscillations. If $f(x)$ is symmetrical with respect to the origin, and is zero except for a range of $(2n+\frac{1}{2})$ wave-lengths within which it is A $\cos\kappa' x$, we have

$$\phi(\kappa) = 2\int_0^{(2n+\frac{1}{2})\pi/\kappa'} \mathrm{A}\cos\kappa'\omega\cos\kappa\omega\,d\omega = 2\kappa'\mathrm{A}\frac{\cos(2n+\frac{1}{2})\pi\kappa/\kappa'}{\kappa'^2-\kappa^2}. \tag{35}$$

Hence, from (25), we have the surface elevation η, of which we write down only the integral necessary for propagation in the direction of x positive, that is,

$$\eta = \frac{\kappa' \mathrm{A}}{\pi} \int_0^\infty \frac{\cos(2n+\frac{1}{2})\,\pi\kappa/\kappa'}{\kappa'^2-\kappa^2} \cos\kappa(x-\mathrm{V}t)\,d\kappa. \tag{36}$$

If n is very large, the main feature consists of the component waves round the value κ'; but in general it is to be noticed that a series of subsidiary components appears whose effects may be of sufficient magnitude to be appreciable. But the component waves are cumulative only for values of x and t such that

$$\kappa = gt^2/4x^2,$$

which is the value corresponding to a stationary argument of the cosine; thus the prominent effect at time t, of any group of parameter κ, will be at localities where κ has the value κ', or, else, a value belonging to one of the subsidiary maxima. The result may be evaluated in the same manner as before; we find

$$\eta = 16\kappa' \mathrm{A} \left(\frac{g}{\pi}\right)^{\frac{1}{2}} \frac{x^{\frac{5}{2}} t}{16\kappa'^2 x^4 - g^2 t^4} \cos(2n+\tfrac{1}{2}) \frac{\pi g t^2}{4\kappa' x^2} \cos\left(\frac{gt^2}{4x} - \tfrac{1}{4}\pi\right). \tag{37}$$

We can obtain the prominent travelling groups above referred to, which this involves, by evaluating the maxima of the amplitude function

$$\frac{t}{16\kappa'^2 x^4 - g^2 t^4} \cos(2n+\tfrac{1}{2}) \frac{\pi g t^2}{4\kappa' x^2}. \tag{38}$$

The form of this function is shown by fig. 1; it is obtained by plotting the curve

$$y = \frac{\alpha}{1-\alpha^4} \cos \tfrac{9}{2}\pi\alpha^2, \tag{39}$$

where α is proportional to t, and, further, α equal to 1 corresponds to κ equal to κ'.

The curve represents the variation of the disturbance at a given point with the time, neglecting the local variations of the last cosine factor in (37); it shows the grouped propagation of an initial displacement consisting of $4\frac{1}{2}$ complete wave-lengths of a simple cosine wave of wave-length $2\pi/\kappa'$, or λ'.

The main undulatory disturbance appears as a simple group around the predominant wave-length λ', moving forward with the corresponding group-velocity $\frac{1}{2}\sqrt{(g/\kappa')}$ or $\frac{1}{2}\mathrm{V}$. But *in advance* of this main group of undulations there are two or three subsidiary groups of sensible magnitude with wave-lengths in the neighbourhood of $9\lambda/2$, $9\lambda/4$, $9\lambda/6$, moving with corresponding group-velocities of $3\mathrm{V}/2\sqrt{2}$, $3\mathrm{V}/4$, $3\mathrm{V}/2\sqrt{6}$. Thus in advance of the main group we have slighter groups of larger wave-lengths moving with group-velocities which may be larger than the wave-velocity of the original disturbance if it were unlimited. In the *rear* of the main group we have also a series of alternating groups, following each other much more quickly and with their wave-lengths and velocities less separated out than in the front of the main group. Hence the disturbance in the rear, especially at distances from the origin not very great, may be expected to consist of small, more irregular, motion resulting from the superposition of this latter system of groups, thus there will be a more distinctive rear of disturbance moving forward with velocity $\frac{1}{2}$V. These inferences may be compared with some results given in Lord Kelvin's later papers. Starting from a solution of the equations for an initial elevation in the form of a single crest, the results were combined graphically so as to show in a series of figures the propagation of

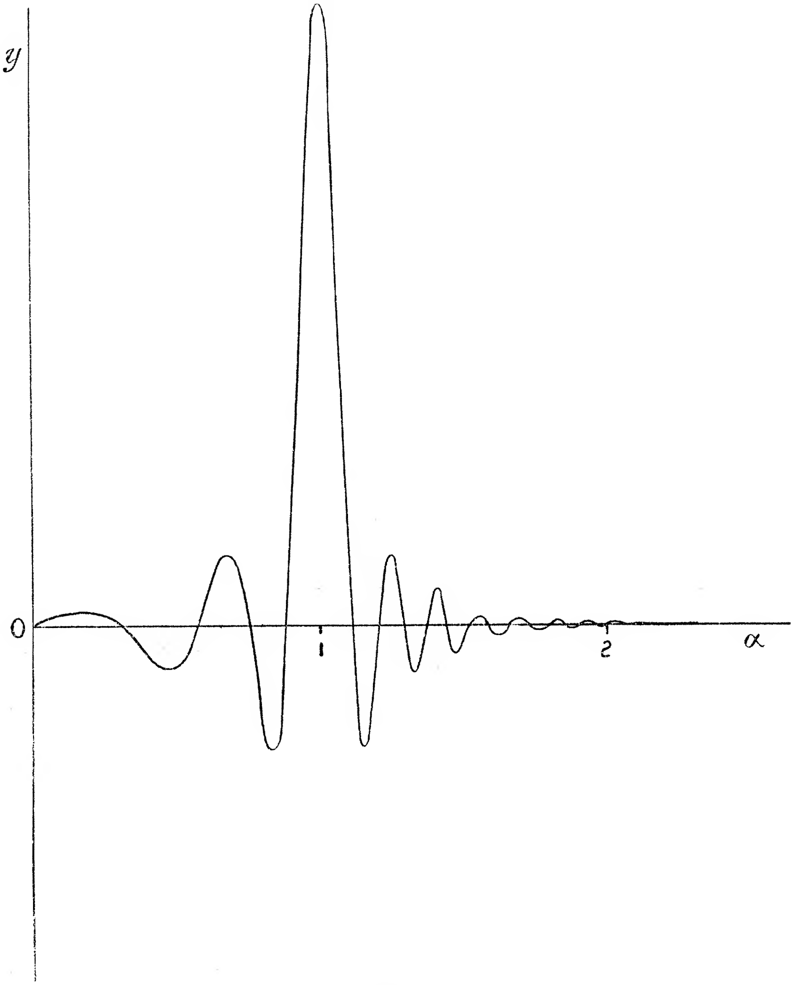

Fig. 1.

an initial disturbance consisting of five crests and four hollows of approximately sinusoidal shape; the following remarks are made:—* "Immediately after the water is left free, the disturbance begins analysing itself into two groups of waves, seen travelling in contrary directions from the middle line of the diagram. The perceptible fronts of these two groups extend rightwards and leftwards from the end of the initial static group far beyond the 'hypothetical fronts,' supposed to travel at half the wave-velocity, which (according to the dynamics of Osborne Reynolds and Rayleigh, in their important and interesting consideration of the work required to feed a uniform procession of water-waves) would be the actual fronts *if* the free groups remained uniform. How far this *if* is from being realised is illustrated by the diagrams of fig. 35, which show a great extension outwards in each direction far beyond distances travelled at half the 'wave-velocity.' While there is this great extension of the fronts outward from the middle, we see that the two groups, after emergence from coexistence in the middle, travel with their rears leaving a widening space between them of water not perceptibly disturbed, but with very minute wavelets in ever augmenting number following slower and slower in the rear of each group. The extreme perceptible rear travels at a speed closely corresponding to the 'half wave-velocity.' . . . . Thus the perceptible front travels at speed actually higher than the wave-velocity, and this perceptible front becomes more and more important relatively to the whole group with the advance of time . . . ."

This extract will serve to emphasise the importance of strict definition and use of the word "group." A simple group, of whatever structure, has associated with it a definite velocity depending only on the wave-length, but not so an arbitrary limited displacement. In various cases we have found it convenient to analyse such into its important elementary groups, each with definite velocity; in special cases the disturbance may be equivalent practically to one simple group.

§ 8. *Initial Impulse on Deep Water.*

Suppose that initially the surface is horizontal, but that given impulses are applied to it. Then for any given symmetrical distribution of impulse $f(x)$, suitable for Fourier analysis, with no initial elevation, the surface elevation at any subsequent time is given by

$$\pi g \rho \eta = \tfrac{1}{2}\int_0^\infty \kappa \mathrm{V} \phi(\kappa) \sin \kappa (x - \mathrm{V}t)\, d\kappa - \tfrac{1}{2}\int_0^\infty \kappa \mathrm{V} \phi(\kappa) \sin \kappa (x + \mathrm{V}t)\, d\kappa, \quad (40)$$

where

$$\phi(\kappa) = \int_{-\infty}^{\infty} f(\omega) \cos \kappa\omega \, d\omega.$$

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* Lord Kelvin, 'Phil. Mag.,' vol. 13, p. 11 (1907).

If we assume $\phi(\kappa)$ equal to 1, so that it is confined to an indefinitely narrow strip of impulse (*cf.* § 5), we obtain the result corresponding to (29) for initial displacement by multiplying that expression by the value of κV; thus we find

$$\eta = \frac{g^{\frac{1}{2}}t^2}{4\pi^{\frac{1}{2}}\rho x^{\frac{5}{2}}}\cos\left(\frac{gt^2}{4x}+\tfrac{1}{4}\pi\right). \tag{41}$$

For comparison with the previous results, suppose that

$$f(x) = \frac{c\alpha^2}{\alpha^2+x^2}, \qquad \phi(\kappa) = \pi c\alpha e^{-\alpha\kappa}.$$

Then we find the surface form as an aggregate of groups, each of them cumulative and so prominent only in a limited region, given by

$$\eta = \frac{\pi^{\frac{1}{2}}c\alpha g^{\frac{1}{2}}t^2}{4\rho x^{\frac{5}{2}}}e^{-\alpha\frac{t^2}{4x^2}}\cos\left(\frac{gt^2}{4x}+\tfrac{1}{4}\pi\right). \tag{42}$$

For a given place the maxima are given by

$$\frac{d}{dt}\left(t^2e^{-g\alpha t^2/4x^2}\right) = 0, \quad \text{that is, by} \quad \frac{x}{t} = \tfrac{1}{2}\sqrt{(g\alpha)}.$$

Thus the maximum moves with velocity $\frac{1}{2}\sqrt{(g\alpha)}$, and consists of nearly simple waves of wave-length $2\pi\alpha$. Comparing with the result in § 6 for an initial displacement of the same character, we see that the maximum is propagated outwards with slower velocity, the wave-length at the maximum being one-half the corresponding value in the former case.

§ 9. *Moving Line Impulse on Deep Water.*

Suppose that the line impulse of the previous section is moving over the surface of deep water at right angles to its length with uniform velocity c, having started at some time practically infinitely remote. Then we may regard the effect at (x, t) as the summation of the effects due to all the consecutive elements of impulse, and we can obtain an expression by modifying (40) and integrating with respect to the time. We measure x from a fixed origin which the line impulse passes at zero time; then we substitute $x-ct_0$ for x and $t-t_0$ for t in (40), and integrate with respect to t_0 for all the time the impulse has been moving. Thus we obtain

$$\begin{aligned}\pi g\rho\eta &= \tfrac{1}{2}\int_{-\infty}^{t}dt_0\int_0^\infty \kappa V\sin\kappa\left\{x-ct_0-V(t-t_0)\right\}d\kappa \\ &\qquad\qquad - \tfrac{1}{2}\int_{-\infty}^{t}dt_0\int_0^\infty \kappa V\sin\kappa\left\{x-ct_0+V(t-t_0)\right\}d\kappa \\ &= \tfrac{1}{2}\int_0^\infty du\int_0^\infty \kappa V\sin\kappa\left\{\varpi+(c-V)u\right\}d\kappa \\ &\qquad\qquad - \tfrac{1}{2}\int_0^\infty du\int_0^\infty \kappa V\sin\kappa\left\{\varpi+(c+V)u\right\}d\kappa,\end{aligned} \tag{43}$$

where $\varpi = x - ct$, and represents distance in advance of the present position of the impulse. We proceed to obtain now the important regular features of the disturbance represented by these integrals.

With the notation of (19) and (20) we have in the first integral

$$f(\kappa) = c - \mathrm{V} = e - \sqrt{(g/\kappa)},$$

$$\frac{d}{d\kappa}\{\kappa f(\kappa)\} = c - \tfrac{1}{2}\sqrt{(g/\kappa)}.$$

Hence the required predominant value of κ, which corresponds to a stationary argument, is given by

$$c - \tfrac{1}{2}\sqrt{\frac{g}{\kappa}} = -\frac{\varpi}{u}, \quad \text{or} \quad \kappa = \frac{gu^2}{4(cu+\varpi)^2}. \tag{44}$$

Thus the first integral in (43) gives

$$\tfrac{1}{4}\pi^{\frac{1}{2}}g^{\frac{3}{2}}\int_0^\infty \frac{u^2}{(\varpi + cu)^{\frac{5}{2}}}\cos\left\{\frac{gu^2}{4(\varpi+cu)} + \tfrac{1}{4}\pi\right\}du. \tag{45}$$

We choose again the principal groups of oscillations by the condition

$$\frac{d}{du}\left\{\frac{gu^2}{4(\varpi+cu)} + \tfrac{1}{4}\pi\right\} = 0, \quad \text{or} \quad cu = -2\varpi.$$

Now u must be positive to come within the range of the integral (45); hence if ϖ is positive we obtain no contribution towards a regular undulatory disturbance. If ϖ is negative we obtain a series of travelling waves which we can evaluate from (45).

We have

$$\frac{d^2}{du^2}\left\{\frac{gu^2}{4(\varpi+cu)} + \tfrac{1}{4}\pi\right\} = -\frac{g}{2\varpi}, \quad \text{when} \quad cu = -2\varpi.$$

Hence, using expression (18), we obtain the value of the chief group from (45), namely,

$$\frac{2\pi g}{c^2}\sin\frac{g\varpi}{c^2}, \tag{46}$$

which holds when ϖ is negative.

As regards the second integral in (43), we easily see by taking the principal group in κ that $\varpi + cu$ must be negative: thus ϖ must be negative and cu between zero and ϖ numerically. Then taking the chief group in u, we have cu equal to 2ϖ numerically. Hence there is no resulting group of waves falling in the range, and the second integral contributes nothing to the regular disturbance.

We have then the well-known result that in front of the travelling impulse there is no regular disturbance, while in the rear there is a train of regular waves, proportional to (46), with wave-length suitable to the velocity c.

The same method can be used for waves on water of depth h, due to a travelling impulse system. For in the integrals (43) we should have

$$f(\kappa) = c - \mathrm{V} = c - \sqrt{\left(\frac{g}{\kappa}\tanh \kappa h\right)}. \tag{47}$$

The group with respect to κ would give a term proportional to

$$\cos\{u\kappa^2 f'(\kappa) + \tfrac{1}{4}\pi\}, \tag{48}$$

where κ has the value given by

$$-\varpi/u = f(\kappa) + \kappa f'(\kappa). \tag{49}$$

We then select the group with respect to u by

$$\frac{d}{du}\{u\kappa^2 f'(\kappa)\} = 0. \tag{50}$$

Using (49) we find this leads to*

$$f(\kappa) = 0, \quad \text{or} \quad \mathrm{V} = c = \sqrt{(gh)}\sqrt{\left(\frac{\tanh \kappa h}{\kappa h}\right)}. \tag{51}$$

Since tanh $\kappa h/\kappa h$ diminishes continually from 1 to 0 as κh increases from 0 to ∞, there is only a real solution of (51) when c^2 is less than gh. In this case we have regular waves of length suitable to velocity c following in the rear of the impulse; when c is greater than the maximum wave-velocity there is no regular wave form.

§ 10. *Capillary Surface Waves.*

In order to illustrate the propagation of an element of the Fourier expression as a limited travelling group of undulations, we consider another form of velocity function. If waves are propagated over the surface of a liquid of density ρ under the action of the surface tension T, it can be shown that the velocity of simple waves of length $2\pi/\kappa$ is

$$\mathrm{V} = \sqrt{(\mathrm{T}\kappa/\rho)}. \tag{52}$$

Hence in this case the group-velocity is

$$\mathrm{U} = \tfrac{3}{2}\sqrt{(\mathrm{T}\kappa/\rho)} = \tfrac{3}{2}\mathrm{V};$$

thus the group-velocity is greater than the wave-velocity, and we shall see how this affects some of the previous results.

(a) *Initial elevation consisting of* $(2n+\frac{1}{2})$ *simple oscillations of wave-length* $2\pi/\kappa'$.—If we consider the same problem as in § 7 we have

$$\eta = \mathrm{A}\int_0^\infty \frac{\cos(2n+\frac{1}{2})\pi\kappa/\kappa'}{\kappa'^2 - \kappa^2}\cos\kappa(x - \mathrm{V}t)\,d\kappa. \tag{53}$$

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* *Cf.* Lord Rayleigh, 'Phil. Mag.,' vol. 10, p. 407 (1905).

The predominant value of κ, for given time and place, is given by

$$\kappa = 4\rho x^2/9\mathrm{T}t^2. \tag{54}$$

The chief groups, each with approximately constant amplitude, are given by

$$\eta = \mathrm{A}' \frac{x^{\frac{1}{2}}t^3}{81\mathrm{T}^2\kappa'^2t^4 - 16\rho^2x^4} \cos\,(2n+\tfrac{1}{2})\frac{4\pi\rho x^2}{9\kappa'\mathrm{T}t^2} \cos\left(\frac{4\rho x^3}{27\mathrm{T}t^3} - \tfrac{1}{4}\pi\right). \tag{55}$$

At a given place the maxima of amplitude are those of

$$\frac{t^3}{81\mathrm{T}^2\kappa'^2t^4 - 16\rho^2x^4} \cos\,(2n+\tfrac{1}{2})\frac{4\pi\rho x^2}{9\kappa'\mathrm{T}t^2}. \tag{56}$$

Fig. 2 represents the curve

$$y = \frac{\alpha^3}{\alpha^4 - 1} \cos\frac{9\pi}{2\alpha^2}, \tag{57}$$

where α is proportional to the time and α equal to 1 corresponds to κ equal to κ'.

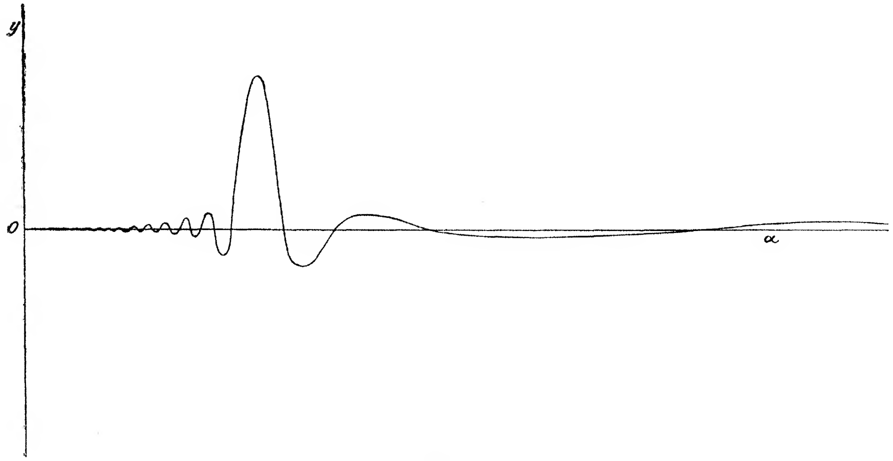

Fig. 2.

Comparing this with § 7 we draw the inference that in this case the perceptible front of the advancing train is more clearly marked than the rear and advances with the half-wave-velocity corresponding to κ', in agreement with simple observation.

(b) *Moving line impulse.*—A line impulse at rest leads to

$$\eta = \mathrm{C}x^{\frac{7}{2}}t^{-4} \cos\left(\frac{4\rho x^3}{27\mathrm{T}t^2} + \tfrac{1}{4}\pi\right).$$

Consequently a moving line impulse will give

$$\eta = \mathrm{A}\int_0^\infty \frac{(\varpi + cu)^{\frac{7}{2}}}{u^4} \cos\left\{\frac{4\rho}{27\mathrm{T}}\frac{(\varpi+cu)^3}{u^2} + \tfrac{1}{4}\pi\right\} du. \tag{58}$$

Then we choose u so that

$$\frac{d}{du}\left\{\frac{(\varpi+cu)^3}{u^2}\right\} = 0, \quad \text{or} \quad u\,(cu-2\varpi)(\varpi+cu)^2 = 0.$$

The value giving a regular wave pattern is the positive root

$$cu = 2\varpi, \text{ for } \varpi \text{ positive.}$$

Hence in this case we have a regular train of waves of length suitable to the velocity c in *advance* of the moving pressure system, with no regular pattern in the rear.

§ 11. *Water Waves due to Gravity and Capillarity.*

If we take account of gravity and the surface tension together, we have the velocity function

$$\mathrm{V} = (\mathrm{T}\kappa+g/\kappa)^{\frac{1}{2}}. \tag{59}$$

Hence $$\mathrm{U} = \frac{d}{d\kappa}(\kappa\mathrm{V}) = \frac{3\mathrm{T}\kappa^2+g}{2\,(\mathrm{T}\kappa^3+g\kappa)^{\frac{1}{2}}}. \tag{60}$$

We have not here a simple ratio U/V, independent of κ. The velocity V has a minimum c_m for a certain value κ_m, equal to $(g/\mathrm{T})^{\frac{1}{2}}$, and for this value U is equal to V—as in fact follows from the definition of U. For $\kappa < \kappa_m$, U is less than V, tending ultimately to $\frac{1}{2}\mathrm{V}$; while for $\kappa > \kappa_m$, U is greater than V and approaches as a limit $\frac{3}{2}\mathrm{V}$.

If we consider a travelling line impulse, the whole problem of finding the principal groups is contained in the equations

$$\left.\begin{aligned} \frac{\varpi+cu}{u} = \mathrm{U} &= \frac{3\mathrm{T}\kappa^2+g}{2\,(\mathrm{T}\kappa^3+g\kappa)^{\frac{1}{2}}} \\ c = \mathrm{V} &= (\mathrm{T}\kappa+g/\kappa)^{\frac{1}{2}} \end{aligned}\right\}. \tag{61}$$

Hence $$c^2u^2 = \frac{4c^4\varpi^2}{c^4-c_m{}^4}, \qquad \kappa = \frac{c^2 \pm (c^4-c_m{}^4)^{\frac{1}{2}}}{2\mathrm{T}},$$

where the positive sign is taken for ϖ positive (in advance of the impulse), and the negative sign for ϖ negative (in the rear). Thus there is no wave pattern unless c is greater than the minimum wave-velocity c_m; and if so there are regular trains both in advance and in the rear, the smaller wave-lengths being in advance. With the ratio c/c_m large, the results approximate to very small waves in front and waves in the rear with κ equal to g/c^2.

§ 12. *Surface Waves in two Dimensions.*

Suppose that the initial data instead of being symmetrical about a transverse straight line are symmetrical around the origin. Let the axes of

x, y be in the undisturbed surface and the axis of z vertically upwards; we write ϖ for $\sqrt{(x^2+y^2)}$. Then, corresponding to (25), the surface elevation ζ due to an initial displacement $f(\varpi)$, set free without initial velocity, is given by

$$\zeta = \int_0^\infty \mathrm{J}_0(\kappa\varpi)\cos(\kappa \mathrm{V}t)\,\phi(\kappa)\,\kappa d\kappa, \tag{62}$$

where

$$\phi(\kappa) = \int_0^\infty f(\alpha)\,\mathrm{J}_0(\kappa\alpha)\,\alpha d\alpha. \tag{63}$$

For an initial point-elevation we may take for simplicity $\phi(\kappa)$ equal to $1/2\pi$; then we have

$$\begin{aligned}\zeta &= \frac{1}{2\pi}\int_0^\infty \mathrm{J}_0(\kappa\varpi)\cos(\kappa \mathrm{V}t)\,\kappa d\kappa \\ &= \frac{1}{\pi^2}\int_0^{\pi/2} d\beta \int_0^\infty \cos(\kappa\varpi\cos\beta)\cos(\kappa \mathrm{V}t)\,\kappa d\kappa \\ &= \frac{1}{2\pi^2}\int_0^{\pi/2} d\beta \int_0^\infty \cos\kappa(\varpi\cos\beta - \mathrm{V}t)\,\kappa d\kappa \\ &\qquad + \frac{1}{2\pi^2}\int_0^{\pi/2} d\beta \int_0^\infty \cos\kappa(\varpi\cos\beta + \mathrm{V}t)\,\kappa d\kappa. \end{aligned} \tag{64}$$

For deep water we separate a real principal group from the first integral, with respect to κ, around the value of κ given by

$$\frac{\varpi\cos\beta}{t} = \tfrac{1}{2}\sqrt{\frac{g}{\kappa}}.$$

This is replaced by the equivalent form

$$\zeta = \frac{g^{\frac{3}{2}}t^3}{8\pi^{\frac{3}{2}}\varpi^{\frac{7}{2}}}\int_0^{\pi/2}\frac{d\beta}{\cos^{\frac{7}{2}}\beta}\cos\left(\frac{gt^2}{4\varpi\cos\beta} - \tfrac{1}{4}\pi\right). \tag{65}$$

Considering now the range for β, we can again select the principal group of oscillations from (65); it occurs at β equal to zero, so we take one-half the result given by the expression (14) and obtain the known result

$$\zeta = \frac{gt^2}{2^{\frac{5}{2}}\pi\varpi^3}\cos\frac{gt^2}{4\varpi}. \tag{66}$$

Similarly, for an initial point impulse we have, instead of (64), the expression

$$\zeta = \frac{1}{2g\rho\pi^2}\int_0^{\pi/2} d\beta \int_0^\infty \kappa \mathrm{V}\{\sin\kappa(\varpi\cos\beta - \mathrm{V}t) - \sin\kappa(\varpi\cos\beta + \mathrm{V}t)\}\kappa d\kappa, \tag{67}$$

leading in the same way to the result

$$\zeta = -\frac{gt^3}{2^{\frac{7}{2}}\pi\rho\varpi^4}\sin\frac{gt^2}{4\varpi}. \tag{68}$$

§ 13. *Point Impulse Travelling over Deep Water.*

Let the impulse be moving along Ox with constant velocity c; let B be the position at time t, A at any previous time t_0, and suppose the system to have been moving for an indefinitely long time.

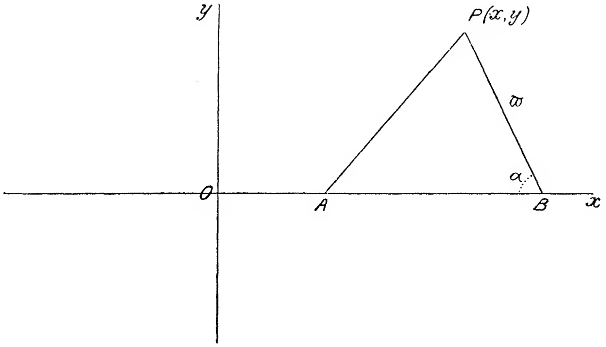

Fig. 3.

We have
$$\mathrm{OA} = ct_0\,;\ \mathrm{OB} = ct\,;$$
$$\mathrm{PB} = \varpi = \{(ct-x)^2+y^2\}^{\frac{1}{2}}\,;$$
$$\cos\alpha = (ct-x)/\varpi.$$

Then in (67) we have to substitute $\{\varpi^2-2c(t-t_0)\cos\alpha+c^2(t-t_0)^2\}^{\frac{1}{2}}$ for ϖ, $t-t_0$ for t, and integrate with respect to t_0 from $-\infty$ to t; we obtain

$$\zeta = \frac{1}{2g\rho\pi^2}\int_0^{\pi/2} d\beta \int_0^{\infty} du \int_0^{\infty} \kappa\mathrm{V}\,[\sin\kappa\,(\cos\beta\,\{\varpi^2-2cu\varpi\cos\alpha+c^2u^2\}^{\frac{1}{2}}-\mathrm{V}u)$$
$$-\sin\kappa\,(\cos\beta\,\{\varpi^2-2cu\varpi\cos\alpha+c^2u^2\}^{\frac{1}{2}}+\mathrm{V}u)]\,\kappa d\kappa. \qquad (69)$$

With $\mathrm{V} = (g/\kappa)^{\frac{1}{2}}$, we select the group around the value of κ given by

$$\kappa^{-1} = 4\cos^2\beta\,(\varpi^2-2cu\varpi\cos\alpha+c^2u^2)/gu^2. \qquad (70)$$

By using the formula (17) we find

$$\zeta = \frac{g^{\frac{3}{2}}}{16\rho\pi^{\frac{3}{2}}}\int_0^{\infty} du \int_0^{\pi/2} \frac{u^4\,d\beta}{\cos^{\frac{9}{2}}\beta\,(\varpi^2-2cu\varpi\cos\alpha+c^2u^2)^{\frac{9}{4}}}$$
$$\cos\left\{\frac{gu^2}{4\cos\beta\,(\varpi^2-2cu\varpi\cos\alpha+c^2u^2)^{\frac{1}{2}}}+\tfrac{1}{4}\pi\right\}. \qquad (71)$$

Selecting from this the chief group which occurs near β equal to zero, we find

$$\zeta = -\frac{g}{2^{\frac{7}{2}}\pi\rho}\int_0^{\infty} \frac{u^3du}{(\varpi^2-2cu\varpi\cos\alpha+c^2u^2)^2}\sin\frac{gu^2}{4\,(\varpi^2-2cu\varpi\cos\alpha+c^2u^2)^{\frac{1}{2}}}. \qquad (72)$$

Finally we choose the chief groups of terms in u from the condition

$$\frac{d}{du}\tfrac{1}{4}gu^2\,(\varpi^2-2cu\varpi\cos\alpha+c^2u^2)^{-\frac{1}{2}} = 0\,; \qquad (73)$$

that is, from
$$c^2u^2-3cu\varpi\cos\alpha+2\varpi^2 = 0, \tag{74}$$
or
$$cu = \tfrac{1}{2}\varpi\left\{3\cos\alpha\pm(9\cos^2\alpha-8)^{\frac{1}{2}}\right\}, \tag{75}$$

We have then different cases to consider according to the nature of these values for cu, remembering that cu gives a position of the moving impulse, at time u previously, for which the waves sent out reinforce each other at the point (ϖ, α) at time t.

(i) In the region where $9\cos^2\alpha < 8$, both roots are imaginary; thus the previous position is non-existent, and there is no principal group in the integral (72). Hence all the regular wave pattern is contained within two straight lines radiating from the point impulse, each making with the line of motion an angle $\cos^{-1} 2\sqrt{2}/3$, or approximately 19° 28′.

(ii) When $9\cos^2\alpha<8$, there are two different real roots for cu. Thus we have two chief groups in the integral (72), corresponding to two regular wave systems superposed on each other.

At any point P within the two bounding radii the disturbance consists of two parts: one part sent out from A at time u_1 previously, where
$$\text{OA} = \tfrac{1}{2}\varpi\left\{3\cos\alpha+(9\cos^2\alpha-8)^{\frac{1}{2}}\right\} \text{ and } u_1 = \text{OA}/c; \tag{76}$$
and another part sent out from B at time u_2 before, where
$$\text{OB} = \tfrac{1}{2}\varpi\left\{3\cos\alpha-(9\cos^2\alpha-8)^{\frac{1}{2}}\right\} \text{ and } u_2 = \text{OB}/c. \tag{77}$$

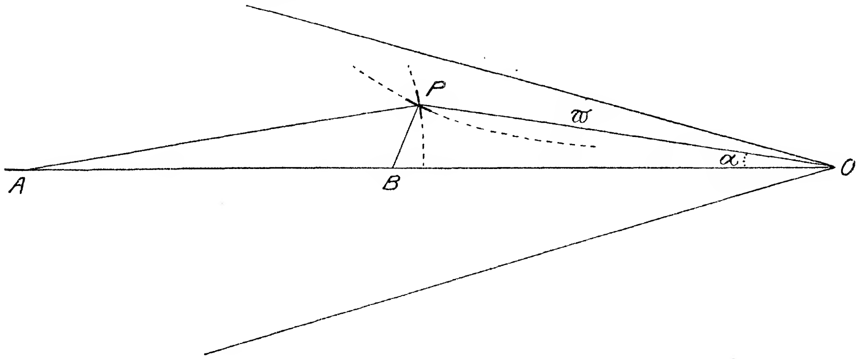

FIG. 4.

We have then two wave systems, which may be called the transverse waves and the diverging waves; we shall examine them separately.

(a) *The transverse wave system.*—Taking the larger value of cu in (76) we find
$$\varpi^2-2cu\varpi\cos\alpha+c^2u^2 = \tfrac{1}{2}\varpi^2\left\{3\cos^2\alpha-2+\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}\right\},$$
$$f(u) = \frac{gu^2}{4\,(\varpi^2-2cu\varpi\cos\alpha+c^2u^2)^{\frac{1}{2}}} = \frac{g\varpi\sqrt{2}}{16c^2}\,\frac{18\cos^2\alpha-8+6\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}}{\left\{3\cos^2\alpha-2+\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}\right\}^{\frac{1}{2}}}. \tag{78}$$

Further, when $f'(u)$ is zero, we have

$$f''(u) = \tfrac{1}{4}gcu\,(2cu-3\varpi\cos\alpha)/(\varpi^2-2cu\varpi\cos\alpha+c^2u^2)^{\frac{3}{2}}. \tag{79}$$

Using the formula (17) we obtain the particular group of terms from the integral (72) as

$$\zeta = -\frac{g}{2^{\frac{7}{2}}\pi\rho}\,\frac{u^3}{(\varpi^2-2cu\varpi\cos\alpha+c^2u^2)^2}\left\{\frac{2\pi}{f''(u)}\right\}^{\frac{1}{2}}\cos\{f(u)-\tfrac{1}{4}\pi\}, \tag{80}$$

in which the special value of u must be substituted.

Evaluating this expression we obtain

$$\zeta = -\frac{g^{\frac{1}{2}}}{2^{\frac{13}{4}}\pi^{\frac{1}{2}}\rho c^3}\,\frac{\{3\cos\alpha+(9\cos^2\alpha-8)^{\frac{1}{2}}\}^{\frac{5}{2}}}{\varpi^{\frac{1}{2}}(9\cos^2\alpha-8)^{\frac{1}{4}}\{3\cos^2\alpha-2+\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}\}^{\frac{5}{4}}}\times$$

$$\cos\left\{\frac{g\varpi\sqrt{2}}{16c^2}\,\frac{\{3\cos\alpha+(9\cos^2\alpha-8)^{\frac{1}{2}}\}^2}{\{3\cos^2\alpha-2+\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}\}^{\frac{1}{2}}}-\tfrac{1}{4}\pi\right\}. \tag{81}$$

This represents a system of transverse waves travelling with the originating impulse; the amplitude for a given azimuth α diminishes as $\varpi^{-\frac{1}{2}}$. On the central line, where α is zero, this reduces to

$$\zeta = -\frac{2^{\frac{1}{2}}g^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\rho c^3\varpi^{\frac{1}{2}}}\cos\left(\frac{g\varpi}{c^2}-\tfrac{1}{4}\pi\right), \tag{82}$$

corresponding to simple line waves of length suitable to velocity c on deep water, but with the amplitude factor $\varpi^{-\frac{1}{2}}$.

Following the crest of a transverse wave we have

$$\frac{g\varpi\sqrt{2}}{16c^2}\,\frac{\{3\cos\alpha+(9\cos^2\alpha-8)^{\frac{1}{2}}\}^2}{\{3\cos^2\alpha-2+\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}\}^{\frac{1}{2}}}-\tfrac{1}{4}\pi = (2n+1)\,\pi, \tag{83}$$

where n is a positive integer. The crests cut the axis in points given by

$$\varpi = c^2(2n+\tfrac{5}{4})\pi/g, \tag{84}$$

and cut the radial boundaries given by $\alpha = \pm\cos 2\sqrt{2}/3$, in the points

$$\varpi = 2c^2(2n+\tfrac{5}{4})\,\pi/g\sqrt{3}. \tag{85}$$

Consider the variation of amplitude following a crest; we substitute for ϖ from (83) in (82) and obtain

$$\zeta = \frac{\text{const.}}{(2n+\frac{5}{4})^{\frac{1}{2}}}\,\frac{\{3\cos\alpha+(9\cos^2\alpha-8)^{\frac{1}{2}}\}^{\frac{7}{2}}}{(9\cos^2\alpha-8)^{\frac{1}{4}}\{3\cos^2\alpha-2+\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}\}^{\frac{3}{2}}}. \tag{86}$$

This becomes infinite at the outer boundary, when α is approximately $19^\circ\,28'$; this is due to the failure of the method of approximation and we shall consider it later. For the present the following table of values and curve show that the approximation holds up to angles very near the limit.

Table I.

ζ = relative amplitude, along the same crest at different azimuths.

α.	ζ.
° ′	
0	1
6	1·03
12	1·18
18	2
19	2·9
19 15	3·5
19 27	7·5
19 28	∞

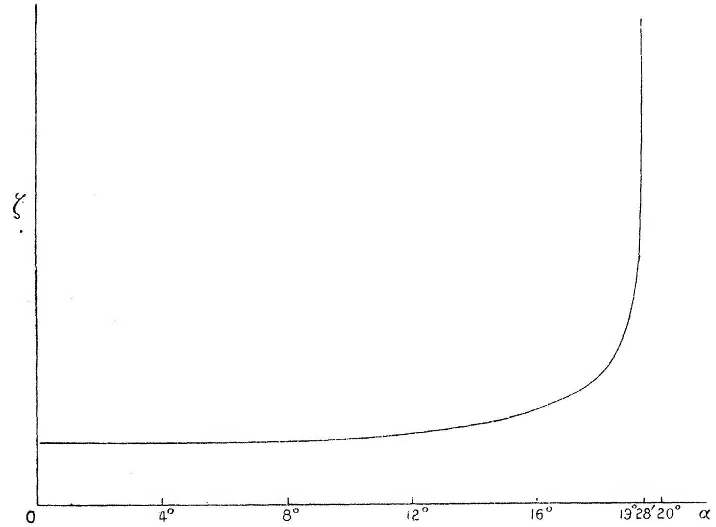

Fig. 5.

(b) *The diverging wave system.*—By taking the smaller root for cu given by (77), we obtain the system of diverging waves; we need only change the sign of the radicle in order to write down the corresponding results in this case.

The crests of the waves are given by

$$\frac{g\varpi\sqrt{2}}{16c^2}\,\frac{\{3\cos\alpha-(9\cos^2\alpha-8)^{\frac{1}{2}}\}^2}{\{3\cos^2\alpha-2-\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}\}^{\frac{1}{2}}}=(2n+\tfrac{5}{4})\,\pi, \tag{87}$$

When α is zero, ϖ is also zero; thus all the crests diverge from the point of impulse. Further, we have

$$\alpha=\cos^{-1}2\sqrt{2}/3\,; \qquad \varpi=2c^2(2n+\tfrac{5}{4})\,\pi/g\sqrt{3}. \tag{88}$$

The law of amplitude along the same crest is given by

$$\zeta = \frac{\text{const.}}{(2n+\frac{5}{4})^{\frac{1}{2}}} \frac{\{3\cos\alpha-(9\cos^2\alpha-8)^{\frac{1}{2}}\}^{\frac{7}{2}}}{(9\cos^2\alpha-8)^{\frac{1}{4}}\{3\cos^2\alpha-2-\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}\}^{\frac{3}{2}}}. \tag{89}$$

In this case, and for the same reason as for the transverse waves, the expression for the amplitude tends to infinity at the outer end of each diverging crest; we shall find an approximation in the next section. But (89) becomes infinitely large for small values of α. From (88) we see that ϖ also becomes small, so that the approximation fails; further, we should expect the expression to become infinite near the impulse on account of its special character. We can show how the infinity disappears if we remove this cause. Consider, as an example, a finite impulse, of constant intensity over a circular area of radius d round the origin, and of zero value outside this circle. Then, as we see from (63), we shall have the same expressions as before, with a new factor given by

$$\begin{aligned}\phi(\kappa) &= \int_0^\infty f(\alpha)\,\mathrm{J}_0(\kappa\alpha)\,\alpha\,d\alpha \\ &= \mathrm{C}\int_0^d \mathrm{J}_0(\kappa\alpha)\,\alpha\,d\alpha = \mathrm{C}\,d\kappa^{-1}\mathrm{J}_1(\kappa d).\end{aligned}$$

Now in the final group for the diverging system we have

$$\kappa = \frac{g}{8c^2}\frac{\{3\cos\alpha-(9\cos^2\alpha-8)^{\frac{1}{2}}\}^2}{3\cos^2\alpha-2-\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}}.$$

Hence the additional factor due to $\phi(\kappa)$ is proportional to

$$\frac{3\cos^2\alpha-2-\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}}{\{3\cos\alpha-(9\cos^2\alpha-8)^{\frac{1}{2}}\}^2}\mathrm{J}_1\left\{\frac{gd\,\{3\cos\alpha-(9\cos^2\alpha-8)^{\frac{1}{2}}\}}{8c^2\,\{3\cos^2\alpha-2-\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}\}}\right\}. \tag{90}$$

When α approaches zero, the argument of the Bessel's function increases indefinitely and we may use the asymptotic expansion; then (90) is proportional to

$$\frac{\{3\cos^2\alpha-2-\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}\}^{\frac{3}{2}}}{\{3\cos\alpha-(9\cos^2\alpha-8)^{\frac{1}{2}}\}^3}. \tag{91}$$

If now we multiply (89) by (91) we obtain a limiting value of the amplitude of the diverging system near the axis; it is proportional to

$$(2n+\tfrac{1}{4})^{-\frac{1}{2}}\{3\cos\alpha-\surd(9\cos^2\alpha-8)\}^{\frac{1}{2}},$$

and the infinity near the axis has disappeared.

(c) *The line of cusps.*—We shall consider now the infinity which occurs at the outer boundary of the two wave systems, when α is $\cos^{-1} 2\surd 2/3$. At any point P the lines of constant phase in the two wave patterns cross at an angle ϕ, which is easily seen to be given by

$$\tan\phi = \tfrac{1}{3}\operatorname{cosec}\alpha\,(9\cos^2\alpha - 8)^{\frac{1}{2}}. \tag{92}$$

As P approaches either radial boundary the two waves ultimately have the same direction, and they will also have the same phase when they meet; consequently an abnormal elevation is to be expected along the two outer boundaries, where the two systems unite in lines of cusps. As we see from (75), the two points A, B coincide for a point P on the line of cusps; and it is on account of this fact that the previous approximations fail for both systems. We have in fact a double root of the equation for finding the chief groups of the integral (72).

Consider the integral

$$y = \int \phi(u)\sin\{f(u)\}\,du, \tag{93}$$

when u_0 is such that

$$f'(u_0) = 0; \qquad f''(u_0) = 0.$$

Following the previous method, we have

$$f(u) = f(u_0) + \tfrac{1}{6}(u-u_0)^3 f'''(u_0);$$

and provided $f'''(u_0)$ is not small, we can write the value of the group for the double root as

$$y = \left\{\frac{6}{f'''(u_0)}\right\}^{\frac{1}{3}} \int_{-\infty}^{\infty} \phi(u_0)\sin\{f(u_0)+\sigma^3\}\,d\sigma \tag{94}$$

$$= \left\{\frac{6}{f'''(u_0)}\right\}^{\frac{1}{3}} \phi(u_0)\sin f(u_0) \int_{-\infty}^{\infty} \cos\sigma^3\,d\sigma.$$

Now at the line of cusps the integral (72) becomes

$$\zeta = -\frac{g}{2^{\frac{7}{2}}\pi\rho}\int_0^{\infty} \frac{u^3\,du}{(\varpi^2 - \frac{4}{3}cu\varpi\sqrt{2} + c^2u^2)^2}\sin\frac{gu^2}{4(\varpi^2 - \frac{4}{3}cu\varpi\sqrt{2} + c^2u^2)^{\frac{1}{2}}}. \tag{95}$$

And we find that $$cu_0 = \varpi\sqrt{2}$$

makes $$f'(u_0) = 0; \qquad f''(u_0) = 0;$$

$$f(u_0) = g\varpi\sqrt{3}/2c^2; \qquad f'''(u_0) = 3gc\sqrt{6}/2\varpi^2.$$

Also we have $$\int_{-\infty}^{\infty}\cos\sigma^3\,d\sigma = \tfrac{2}{3}\pi/\Gamma(\tfrac{2}{3}).$$

Hence, substituting these values, we have

$$\zeta = -\frac{3^{\frac{5}{6}}g^{\frac{2}{3}}}{2^{\frac{1}{2}}c^{\frac{10}{3}}\rho\Gamma(\frac{2}{3})\varpi^{\frac{1}{3}}}\sin\frac{g\varpi\sqrt{3}}{2c^2}. \tag{96}$$

We notice first the difference of phase of $\frac{1}{4}\pi$ between this and the expressions for the separate systems where they cut the outer boundaries; this is analogous to the change of phase along an optical ray in passing a focus. We saw that the separate transverse and diverging crests converged towards points of equal phase on the outer boundaries given by

$$\varpi = 2c^2(2n+\tfrac{5}{4})\,\pi/g\sqrt{3},$$

but with the result given in (96) we see that the actual crests on the line of cusps are given by

$$\varpi = 2c^2(2n+\tfrac{3}{2})\,\pi/g\sqrt{3}. \tag{97}$$

The amplitude of the cusped waves diminishes at a slower rate than the transverse waves, so that their size becomes relatively more marked towards the rear of the disturbance. The amplitude of successive crests is given by (96) and (97) as

$$\zeta_{mc} = \frac{3}{2^{\frac{5}{6}}\Gamma(\frac{2}{3})(2n+\frac{3}{2})^{\frac{1}{3}}\pi^{\frac{1}{3}}}\frac{g}{c^4\rho}. \tag{98}$$

The amplitude of successive crests of the transverse waves where they cut the axis are given by (82) and (84), and we find

$$\zeta_{ma} = \frac{2^{\frac{1}{2}}}{(2n+\frac{5}{4})^{\frac{1}{2}}\pi}\frac{g}{c^4\rho}. \tag{99}$$

Taking the ratio of these two quantities we have an expression for the magnitude of the crests at the cusps compared with the transverse crests on the axis; approximately

$$\frac{\zeta_{mc}}{\zeta_{ma}} = 1{\cdot}9\,\frac{(2n+\frac{5}{4})}{(2n+\frac{3}{2})^{\frac{1}{3}}}. \tag{100}$$

The following table and curve show how the successive crests at the axis and outer line diminish, and exhibit their relative magnitudes for different values of n.*

---

* On August 3, 1887, Lord Kelvin delivered a lecture "On Ship Waves" before the Institution of Mechanical Engineers at Edinburgh, in which he appears to have shown a model to scale of the theoretical wave pattern produced by a ship. Only a diagram of the crest curves has been published ('Popular Lectures,' vol. 3, p. 482); the form of the crests agrees with that deduced above, except of course near the disturbance or the radial boundaries. It has, in fact, been verified that on substituting his expressions for x, y in terms of a parameter w in the present equations, the latter are satisfied identically. The law of amplitude along the waves is not stated by Lord Kelvin; as Prof. Lamb conjectures, his result seems to have been obtained by an application of the idea of group-velocity (H. Lamb, 'Hydrodynamics,' § 253, 1906).

Table II.

n.	ζ_{ma}.	ζ_{mc}.	ζ_{mc}/ζ_{ma}.
5	15	35	2·3
10	10·8	28·6	2·6
15	9	25	2·7
20	7·7	23	3
50	5	17	3·4
100	3·6	13·6	3·8

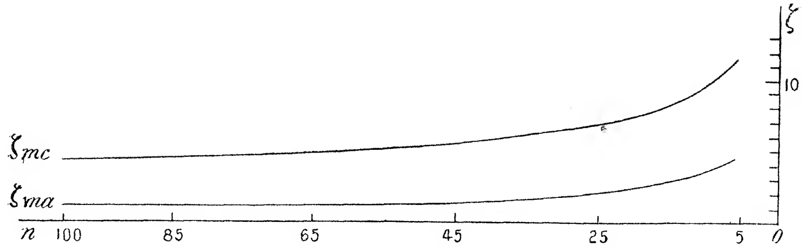

Fig. 6.

§ 14. *Point Impulse for Different Media.*

Consider a point impulse moving with uniform velocity c over the surface of a dispersive medium for which U and V are respectively the group- and wave-velocity for a value κ of $2\pi/\lambda$.

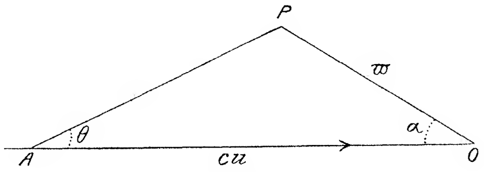

Fig. 7.

Let the disturbance from the impulse when in the neighbourhood of a point A combine so as to produce waves κ at P at the present moment when the impulse is at O. Then the problem of finding the possible persistent wave systems is contained in the equations

$$\frac{\mathrm{AP}}{\mathrm{AO}} = \frac{\mathrm{U}}{c}, \qquad c\cos\theta = \mathrm{V}; \tag{101}$$

that is, in

$$(\varpi^2 - 2cu\varpi\cos\alpha + c^2u^2)^{\frac{1}{2}}/u = \mathrm{U}, \tag{102}$$

$$c\,(cu - \varpi\cos\alpha)/(\varpi^2 - 2cu\varpi\cos\alpha + c^2u^2)^{\frac{1}{2}} = \mathrm{V}. \tag{103}$$

The wave pattern depends upon the character of the positive roots of these equations for cu and κ; each such value of cu defines a wave system with wave front through P at right angles to AP, and each system can be expressed in the form

$$\zeta = \mathrm{F}(\varpi, \alpha) \cos \{\kappa(\varpi^2 - 2cu\varpi \cos \alpha + c^2u^2)^{\frac{1}{2}} + \epsilon\},$$

with cu and κ as functions of ϖ and α.

Suppose the medium is such that the group-velocity bears a constant ratio to the wave-velocity, that is, suppose

$$\mathrm{U} = \tfrac{1}{2}(n+1)\,\mathrm{V}, \tag{104}$$

where n is independent of κ.

Then the equations (102) and (103) lead to a quadratic for cu, namely,

$$(1-n)\,c^2u^2 + (n-3)\,cu\varpi \cos \alpha + 2\varpi^2 = 0. \tag{105}$$

Hence we have the roots

$$cu = \frac{\varpi}{2(1-n)}[(3-n)\cos\alpha \pm \sqrt{\{(3-n)^2\cos^2\alpha - 8(1-n)\}}]. \tag{106}$$

We shall examine some special cases.

(*a*) $0<n<1$.—There are two positive values of cu which are real, provided

$$\cos^2\alpha > 8(1-n)/(3-n)^2.$$

Thus there are two wave systems, transverse and diverging, with a line of cusps corresponding to the double roots, and the whole wave pattern is included within an angle

$$2\cos^{-1}\{8(1-n)\}^{\frac{1}{2}}/(3-n), \tag{107}$$

which increases with n.

The previous section on deep-water waves is the case of n zero.

(*b*) $n = 1$. This is a critical case, implying coincidence of wave-velocity with group-velocity, and consequently no dispersion.

(*c*) $n = 2$. This is the case of capillary surface waves. We see that there is only one positive root of the quadratic, and it is real for all values of α; the root is

$$cu = \tfrac{1}{2}\varpi\,\{(\cos^2\alpha + 8)^{\frac{1}{2}} - \cos\alpha\}. \tag{108}$$

There is only one wave system, but it extends over the whole surface; along the line of motion κ is zero in the rear, while in advance of the impulse it is of value suitable to simple waves moving with velocity c.

(*d*) $n = 3$. This holds for flexural waves on a plate; there is one system of waves extending over the surface, corresponding to the root $cu = \varpi$.

The crests, and other lines of equal phase, are given by the curves

$$\varpi \sin^3 \tfrac{1}{2}\alpha = \text{constant}.$$

(*e*) *Gravity and capillarity combined.*—The relation between U and V is not a constant ratio in this case; we had in § 11 the expressions for the two velocities as functions of κ. It can be shown that in certain cases the equations for cu lead to four possible roots, giving four wave-branches through the point.

§ 15. *Point Impulse moving on Water of Finite Depth.*

With the same problem we have now, if the water is of depth h,

$$V = \left(\frac{g}{\kappa}\tanh \kappa h\right)^{\frac{1}{2}},$$

$$U = \tfrac{1}{2}\left(\frac{g}{\kappa}\tanh \kappa h\right)^{\frac{1}{2}}.\left(1+\frac{2\kappa h}{\sinh 2\kappa h}\right). \tag{109}$$

If we write

$$U = \tfrac{1}{2}(n+1)\,V,$$

n varies between 0 and 1, being dependent upon the value of κ. We use the notation

$$p = \frac{gh}{c^2}, \qquad m = \frac{\tanh \kappa h}{\kappa h}, \qquad n = \frac{2\kappa h}{\sinh 2\kappa h}. \tag{110}$$

Then m and n are monotonic functions of κ with the following limiting values:

$$\kappa = 0; \quad m = 1; \quad n = 1.$$
$$\kappa = \infty; \quad m = 0; \quad n = 0.$$

The two equations for cu and κ become

$$\frac{(\varpi^2 - 2cu\varpi \cos\alpha + c^2u^2)^{\frac{1}{2}}}{cu} = \tfrac{1}{2}(pm)^{\frac{1}{2}}(1+n), \tag{111}$$

$$\frac{cu - \varpi\cos\alpha}{(\varpi^2 - 2cu\varpi\cos\alpha + c^2u^2)^{\frac{1}{2}}} = (pm)^{\frac{1}{2}}. \tag{112}$$

From these we obtain

$$\cos^2\alpha = \frac{\{1-\frac{1}{2}pm(1+n)\}^2}{1-\frac{1}{4}pm(1+n)(3-n)}; \tag{113}$$

$$cu = \frac{\varpi}{2(1-n)}\left[(3-n)\cos\alpha \pm \{(3-n)^2\cos^2\alpha - 8(1-n)\}^{\frac{1}{2}}\right]. \tag{114}$$

Combining the last two we have the values of cu as

$$cu = \frac{2\varpi}{1-n}\{1-\tfrac{1}{4}pm(1+n)(3-n)\}^{\frac{1}{2}}, \tag{115}$$

or

$$cu = \varpi/\{1-\tfrac{1}{4}pm(1+n)(3-n)\}^{\frac{1}{2}}. \tag{116}$$

We have two cases to consider according as $p >$ or < 1.

(*a*) $c<\sqrt{(gh)}$; $p>1$.—From (114) we see that the equal values of cu, defining the lines of cusps within which the wave pattern lies, are given by such values of κ that

$$\cos^2\alpha = \frac{8(1-n)}{(3-n)^2}. \tag{117}$$

Whatever the value of κ, n can only lie between 1 and 0; hence α can only lie between $\cos^{-1}2\sqrt{2}/3$ and $\pi/2$, or between 19° 28′ and 90°. The smaller value is the limiting angle for deep water, when n is considered zero for all values of κ.

We see from (115) and (116) that the equal values of cu occur when

$$1-\tfrac{1}{4}pm(1+n)(3-n) = \tfrac{1}{2}(1-n),$$

or when

$$m(3-n) = 2/p.$$

The greatest possible value of $m(3-n)$ is 2; hence we have the limitation $p>1$. Only in this case is there a double wave system with a line of cusps.

As p decreases to 1, that is as the velocity c approaches the critical value $\sqrt{(gh)}$, m and n at the line of cusps both approach their limiting value 1; and at the same time the cusp angle widens out, approaching a right angle. Further, along the axis we have

$$pm = 1, \qquad m = 1/p = c^2/gh.$$

Hence on the axis the transverse waves are the simple waves travelling with velocity c on water of depth h. As p decreases to 1, the wave-length increases indefinitely; m, and consequently n, approach unity on the axis.

Now if n is 1, the group-velocity U equals the wave-velocity V, and the medium is non-dispersive. Thus at the critical velocity c, equal to $\sqrt{(gh)}$, we have a source emitting disturbances and travelling at the rate of propagation of the disturbances; we see that the whole effect is practically concentrated into a line through the source at right angles to the direction of motion. This agrees with observations of ship waves when approaching shallow water at the critical velocity.*

(*b*) $c>\sqrt{(gh)}$; $p<1$.—We may now have the greatest value, unity, of m; it is easily seen that for less values of m and n the values of α given by (113) become smaller.

At the outer limit we have

$$\cos^2\alpha = 1-p, \qquad \sin^2\alpha = p = gh/c^2. \tag{118}$$

Consequently the wave pattern is contained within two lines making with the axis an angle which diminishes as c increases.

---

* 'Trans. Inst. Nav. Arch.,' vol. 47, p. 353 (1905). Compare also the motion of an electron with the velocity of radiation.

Further, since equal values of cu are given by

$$m(3-n) = 2/p,$$

we see that there are no cusps, for the left-hand side cannot be greater than 2.

The values of cu given in (115) and (116) correspond to the transverse and diverging waves respectively. If we substitute (116) in equations (111) and (112) we find that they are satisfied identically; hence there is always a diverging wave system. On the other hand, if we substitute (115) we find we must have

$$pm = \frac{1-pm}{1-\frac{1}{4}pm(1+n)(3-n)}, \quad \text{or} \quad m(2-n) = \frac{1}{p}.$$

But the greatest possible value of the left-hand side is unity.

Hence there can be a transverse wave system only so long as p is greater than 1; when c exceeds $\sqrt{(gh)}$, the transverse waves disappear.

At the outer line given by

$$\sin^2\alpha = p, \qquad m = n = 1,$$

we have, for the diverging waves,

$$cu = \varpi(1-p)^{-\frac{1}{2}} = \varpi \sec\alpha.$$

Hence the outer line forms a wave front of the diverging wave system. We see also that the other wave fronts (lines of equal phase) are now concave to the axis, instead of being convex as when $p > 1$. There is no definite inner limit to the system; as the axis is approached, the wave fronts become more nearly parallel to the axis, and the wave-length diminishes indefinitely. Finally, as the velocity c is increased, the angle α diminishes, and the regular waves are contained within a narrower angle radiating from the centre of disturbance.

The following tables (III) and (IV) and the curve in fig. 8 show how the angle α varies as the velocity c is increased up to and beyond the critical velocity.

Table III.

κh at cusps.	p.	α.		$c/\sqrt{(gh)}$.
		°	′	
10	7	19	28	0·38
8	5·4	19	28	0·42
6	4	19	29	0·5
5	3·3	19	30	0·55
4	2·7	19	37	0·6
3	2	20	18	0·7
2	1·5	23	42	0·82
1	1·18	39	19	0·92
0·5	1·08	59	27	0·96
0·2	1·01	78		0·99
0	1	90		1

$\alpha = \cos^{-1}\sqrt{8(1-n)}/(3-n)$.

Table IV.

p.	$\alpha = \sin^{-1}\surd(p)$.	$c/\surd(gh)$.
	° ′	
0·99	84	1·005
0·5	45	1·41
0·33	35	1·73
0·25	30	2
0·11	19 28	3

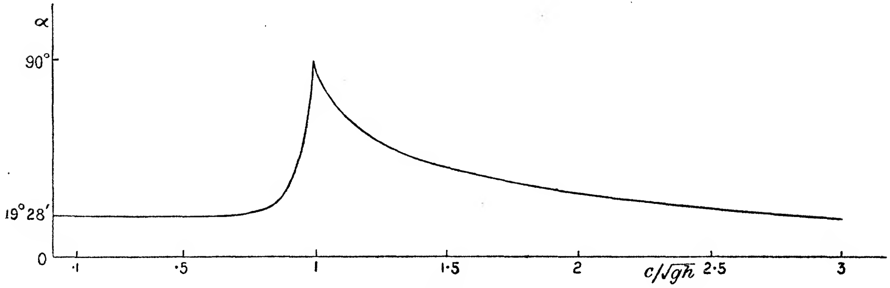

Fig. 8.

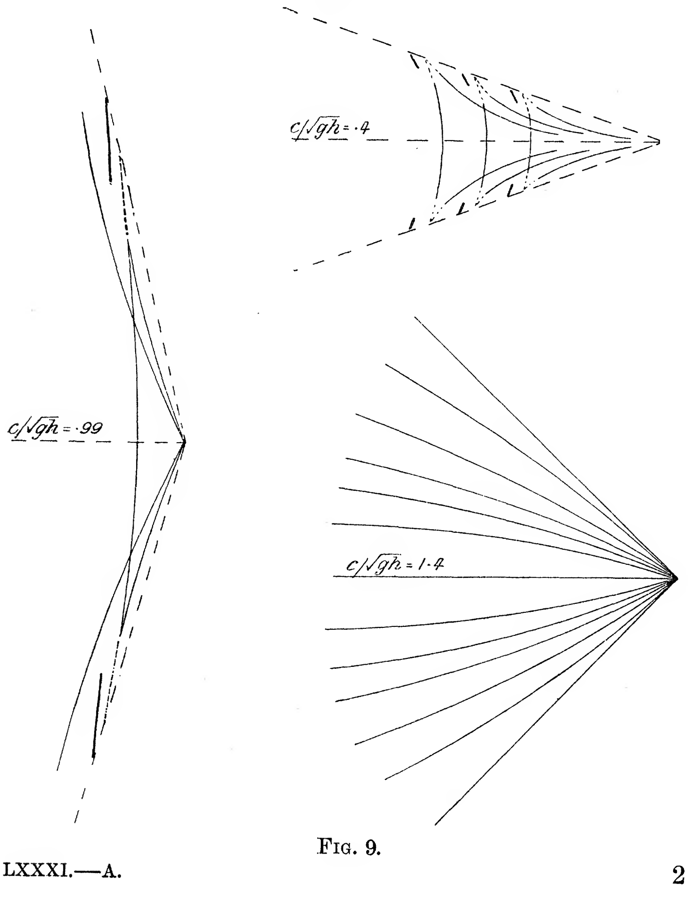

Fig. 9.

With the help of these results, sketches are given in fig. 9 to represent the change in the wave pattern, as the critical velocity is approached and passed.

On the Generation of a Luminous Glow in an Exhausted Receiver moving near an Electrostatic Field, and the Action of a Magnetic Field on the Glow so produced; the Residual Gases being Oxygen, Hydrogen, Neon, and Air.—Part 3.

By Frederick John Jervis-Smith, M.A. (Oxon), F.R.S.

(Received September 9,—Read November 5, 1908.)

(1) A silica bulb, similar to those employed in the experiments described at p. 214, 'Roy. Soc. Proc.,' A, vol. 81, was rotated four to five revolutions per second; the degree of exhaustion was similar to that reached in the former experiments, the residual gas being oxygen. The inductor was charged gradually, until the bulb glowed; then slowly discharged through a piece of damped thread, until the glow entirely died out; the graduated electroscope being observed during the slow discharge. On establishing the magnetic field, the brilliant glow was at once restored. This phenomenon could be repeated at any time with certainty. In some experiments the south pole was effective, the north pole not so. This was probably connected with the fact that the north pole deflected the glow away from the stem of the bulb, which was the axis of rotation, and thus in contact with outside bodies.

(2) A silica bulb, the residual gas being air, when similarly treated, glowed; but the magnetic effect was less marked than when oxygen was the residual gas.

(3) A bulb of *glass* was next rotated; the inductor being charged as in the former experiment; the glow was very much less than when a silica bulb was used. The glow was deflected by the magnetic field, but not appreciably increased. Even when the inductor was charged to 3000 volts the glow was feeble.

(4) In order to discover how far the residual gas in the exhausted bulb affected the phenomena, I determined to contrast the effects produced in silica bulbs when neon or oxygen were the residual gases in the bulbs.

Sir William Ramsay very kindly offered to prepare me a bulb, in which the residual gas was neon.